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About uniformly Menger spaces

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Abstract. Precompact type properties – precompactness (=totally precompactness), σ-precompactness, pre-Lindelöfness, (=$\aleph_0$-boundedness), τ-boundedness – belong to the basic important invariants studied in the uniform topology.

The theory of these invariants is wide and continues to develop. However, in a sense, the class of uniformly Menger spaces escaped the attention of researchers.

Lj.D.R. Kočinac was the first who introduced and studied the class of uniformly Menger spaces in [3, 4]. It immediately follows from the definition that the class of uniformly Menger spaces lies between the class of precompact uniform spaces and the class of pre-Lindelöf uniform spaces. Therefore, we expect it to have many good properties.

In this paper some important properties of the uniformly Menger spaces are investigated. In particular, it is established that under uniformly perfect mappings, the uniformly Menger property is preserved both in the image and the preimage direction.

1. Introduction

Throughout this paper all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.

For covers α and β of a set X, we have:

$$\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, \ \ B \in \beta\},$$

$$\alpha(x) = \bigcup St(\alpha, x), \ \ St(\alpha, x) = \{A \in \alpha : x \in A\}, \ \ x \in X,$$

$$\alpha(H) = \bigcup St(\alpha, H),$$

$$St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \varnothing\}, \ \ H \subset X.$$

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For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha* \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$.

A *uniformity* on a nonempty set X is a family U of covers of X which satisfies the following conditions:

(U1) if $\alpha \in U$ and β is a cover of X such that $\alpha \succ \beta$, then $\beta \in U$;
(U2) if $\alpha_1, \alpha_2 \in U$, then there exists $\alpha \in U$ such that $\alpha \succ \alpha_1$ and $\alpha \succ \alpha_2$;
(U3) if $\alpha \in U$, then there exists $\beta \in U$ such that $\beta* \succ \alpha$;
(U4) for any two distinct points x and y in X there exists an $\alpha \in U$ such that no member of α contains both x and y.

The covers from U are called *uniform covers*, and the pair (X, U) a *uniform space.*

If U_1 and U_2 are two uniformities on a set X and $U_2 \subset U_1$, then we say that the uniformity U_1 is finer than the uniformity U_2 or that U_2 is coarser than U_1.

A uniform space (X, U) is called:

(1) *precompact*, if the uniformity U has a base consisting of finite covers [1];
(2) *σ-precompact* (*σ-compact*), if it can be represented as the union of countably many precompact (*compact*) subspaces [1];
(3) *totally bounded,* if each $\alpha \in U$ there is a finite set $M \subset X$ such that $\alpha(\mathrm{M}) = \mathrm{X}$ [1];
(4) *pre-Lindelöf* or *$\aleph_0$-bounded*, if the uniformity U has a base consisting of countable covers [1, 3, 5];
(5) *uniformly locally compact*, if the uniformity U contains a uniform cover consisting of compact sets [2];
(6) *uniformly Menger space* or has the *uniform Menger property*, if for each sequence $(\alpha_n : n \in \mathbb{N}) \subset U$ there is a sequence $(\beta_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, β_n is a finite subset of α_n and $\bigcup\limits_{n\in\mathbb{N}} \beta_n$ is a cover of X [3, 4].

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). The mapping f is called:

(1) *precompact*, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and a finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [1];
(2) *uniformly perfect*, if it is both precompact and perfect [1].

A cover α of uniform space (X, U) is called *co-cover*, if $\alpha \bigcap F \neq \varnothing$ for all free Cauchy filters F of (X, U). A Cauchy filter F of (X, U) is called a free Cauchy filter, if $\bigcap\{[M] : M \in F\} = \varnothing$, where $[M]$ is the closure of the set M. For a uniformity U by τ_U we denote the topology generated by the uniformity. The least upper bound of all uniformities on a Tychonoff space X, i.e., the finest uniformity on the space X, is called the *universal uniformity* on the space X and is denoted by U_X.

2. Uniformly Menger spaces

The theory of selection principles has been developing intensively. A search for uniform analogues of the basic concepts and statements of the theory of selection principles is an actual task in the uniform topology. For the first time, some uniform analogues of the theory of selection principles have been studied in the work of the Serbian mathematician Lj.D.R. Kočinac [3].

This paper studies some properties of uniformly Menger spaces.

We need the following simple uniform topological lemma.

Lemma 1. *A uniform space (X, U) is pre-Lindelöf if and only if each uniform cover $\alpha \in U$ contains a countable subcover $\alpha_0 \subset \alpha$.*

Proof. Let $\alpha \in U$ be an arbitrary uniform cover. Since (X, U) is pre-Lindelöf, there is a countable uniform cover $\beta \in U$ such that $\beta \succ \alpha$. For any $B_n \in \beta$ we choose one $A_{B_n} \in \alpha$ such that $B_n \subset A_{B_n}$. Put $\alpha_0 = \{A_{B_n} : n \in \mathbb{N}\}$. Then $\alpha_0 \subset \alpha$ is a countable subcover.

Conversely, let $\alpha \in U$ be an arbitrary uniform cover and $\beta, \gamma \in U$ be two uniform covers such that $\gamma * \succ \beta$ and $\beta * \succ \alpha$. Let γ_0 be a countable subcover of γ. For each $\Gamma \in \gamma_0$ choose one element $x_\Gamma \in \Gamma$ and put $\mathrm{M} = \{x_\Gamma : \Gamma \in \gamma_0\}$. It is clear that M is a countable subset of the space (X, U). Let $\Gamma \in \gamma_0$ be an arbitrary element and $y \in \Gamma$ be an arbitrary selected point. Then there is $x_\Gamma \in \mathrm{M}$ such that $y \in \gamma(x_\Gamma)$. There is $B \in \beta$ such that $x_\Gamma \in \gamma(y) \subset \gamma(\Gamma) \subset B$. It follows from this, that $\gamma(\Gamma) \subset \beta(x_\Gamma)$. Now for x_Γ choose one $A_{x_\Gamma} \in \alpha$ such that $\beta(x_\Gamma) \subset A_{x_\Gamma}$. Let $\alpha_0 = \{A_{x_\Gamma} : x_\Gamma \in \mathrm{M}\}$. Then $\gamma * \succ \alpha_0$. Therefore, $\alpha_0 \in U$. So (X, U) is pre-Lindelöf. $\square$

Theorem 1 ([3])**.** *Any precompact uniform space (X, U) is a uniformly Menger space, and any uniformly Menger space (X, U) is pre-Lindelöf.*

Proof. Let (X, U) be a precompact space and $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence. Then, for any $n \in \mathbb{N}$ the cover α_n contains a finite subcover $\alpha_n^0 \subset \alpha_n$. Put $(\beta_n : n \in \mathbb{N})$, $\beta_n = \alpha_n^0$. Then $(\beta_n : n \in \mathbb{N})$ is the desired sequence of finite subfamilies. Hence, (X, U) is a uniformly Menger space.

Let now $\alpha \in U$ be an arbitrary uniform cover of a uniformly Menger space (X, U). Put $\alpha_n = \alpha$ for any $n \in \mathbb{N}$. Then, for the sequence $(\alpha_n : n \in \mathbb{N}) \subset U$, where $\alpha_n = \alpha$, $n \in \mathbb{N}$, there is a sequence $(\beta_n : n \in \mathbb{N})$ of finite subfamilies such that for any $n \in \mathbb{N}$, β_n is a subfamily of the cover α_n, i.e. α and $\bigcup\limits_{n \in \mathbb{N}} \beta_n$ is a cover of the space (X, U). For each $n \in \mathbb{N}$ and for each element $\mathrm{B}_{\beta_n(i)} \in \beta_n$, $i = 1, 2, \ldots, k$ by selecting one element $\mathrm{A}_{\mathrm{B}_{\beta_n(i)}}$ from $\alpha = \alpha_n$, we obtain a finite subfamily $\alpha_n^0 \subset \alpha$. Then $\bigcup\limits_{n\in\mathbb{N}} \alpha_n^0$ is a countable subfamily of the cover α. Since the family $\bigcup\limits_{n\in\mathbb{N}} \beta_n$ is a cover of the space (X, U), the family $\bigcup\limits_{n\in\mathbb{N}} \alpha_n^0$ is also a cover of (X, U). According to Lemma 1, the space (X, U) is a pre-Lindelöf space. $\square$

Theorem 2. *Any σ-precompact uniform space (X, U) is a uniformly Menger space.*

Proof. Let (X, U) be a σ-precompact space and $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence, $X = \bigcup\limits_{n\in N} X_n$, where each X_n is precompact. Then, for any $n \in \mathbb{N}$, the cover $\alpha_{X_n} = \alpha_n \wedge \{X_n\}$ of X_n contains a finite subcover $\alpha_{X_n}^0 \subset \alpha_{X_n}$. Put $\beta_{X_n} = \alpha_{X_n}^0$. Then, for each $n \in \mathbb{N}$, β_{X_n} is a finite subset of α_n and $\bigcup\limits_{n\in\mathbb{N}} \beta_{X_n}$ is a cover of X. Hence (X, U) is a uniformly Menger space. $\square$

Corollary 1. *Any σ-compact uniform space (X, U) is a uniformly Menger space.*

Corollary 2. *Any σ-compact Tychonoff space X is a Menger space.*

Theorem 3 ([3])**.** *A Tychonoff space X is a Menger space if and only if the uniform space (X, U_X), where U_X is the universal uniformity, is a uniformly Menger space.*

Proof. Let X be a Menger space and $(\alpha_n : n \in \mathbb{N}) \subset U_X$ be an arbitrary sequence of uniform covers. Since the interior $\langle\alpha_n\rangle$ of each uniform cover α_n is an open cover, then $\{\langle\alpha_n\rangle\}$ is a sequence of open covers of the spaces X, $\langle\alpha_n\rangle = \{\langle \mathrm{A}\rangle : \mathrm{A} \in \alpha_n\}$, where $\langle \mathrm{A}\rangle$ is the interior of the set A. Then, there is a sequence $(\beta_n : n \in \mathbb{N})$ of finite open subfamilies such that for any $n \in \mathbb{N}$, β_n is a finite subfamily for α_n and $\bigcup\limits_{n\in\mathbb{N}} \beta_n$ is an open cover of X. Consequently, (X, U_X) is a uniformly Menger space.

Conversely, let $(\alpha_n : n \in \mathbb{N})$ be an arbitrary sequence of open covers of the space X. Then $(\alpha_n : n \in \mathbb{N}) \subset U_X$. Therefore, there is a sequence $(\beta_n : n \in \mathbb{N})$ of finite families such that for any $n \in \mathbb{N}$, the family β_n is a subfamily of α_n and $\bigcup\limits_{n\in\mathbb{N}} \beta_n$ is a cover of the space (X, U_X). Assume $\gamma_n = \langle\beta_n\rangle$, $\langle\beta_n\rangle = \{\langle \mathrm{B}\rangle : \mathrm{B} \in \beta_n\}$, where $\langle \mathrm{B}\rangle$ is the interior of the set B. Note that $(\gamma_n : n \in \mathbb{N})$ is a sequence of finite subfamilies and for any $n \in \mathbb{N}$ the family γ_n is a subfamily of α_n and the family $\bigcup\limits_{n\in\mathbb{N}} \gamma_n$ is an open cover of the spaces X. Consequently, X is a Menger space. $\square$

Theorem 4. *Let $f : (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) onto a uniform space (Y, V). If (Y, V) is a uniformly Menger space, then (X, U) is also a uniformly Menger space.*

Proof. Let $f : (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) onto a uniformly Menger space (Y, V) and $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers. Then, for any $n \in \mathbb{N}$, there exists a finite cover $\gamma_n \in U$ and $\beta_n \in V$, such that $f^{-1}\beta_n \wedge \gamma_n \succ \alpha_n$. Since (Y, V) is a uniformly Menger space, then for the sequence $(\beta_n : n \in \mathbb{N}) \subset V$ there exists a sequence $\{\beta_n^o\}$ of finite subfamilies such that $\bigcup\limits_{n\in\mathbb{N}} \beta_n^o$ is a cover of the space (Y, V). Note, that for any $n \in \mathbb{N}$ the family $f^{-1}\beta_n^o \wedge \gamma_n$ is finite and in addition $\bigcup\{f^{-1}\beta_n^o \wedge \gamma_n\} = \bigcup f^{-1}\beta_n^o$. Next, for any $f^{-1}B_{n,i}^o \bigcap \Gamma_{n,i} \in f^{-1}\beta_o \wedge \gamma_n$ choose $A_{n,i}^o \in \alpha_n$ such that $f^{-1}B_{n,i}^o \bigcap \Gamma_{n,i} \subset A_{n,i}^o$. Put $\alpha_n^o = \{A_{n,1}^o, A_{n,2}^o, \dots, A_{n,k}^o\}$, where k is the cardinality of $f^{-1}\beta_n^0 \wedge \gamma_n$. It is easy to see that the family $\bigcup\limits_{n\in\mathbb{N}} \alpha_n^o$ is a cover of the space (X, U). Therefore, (X, U) is a uniformly Menger space. □

From [3, Theorem 6] and Theorem 4, the following theorem takes place.

Theorem 5. *Let $f : (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) onto a uniform space (Y, V). Then, uniformly Menger's property is preserved both in the image and the preimage direction.*

Corollary 3. *Let $f : (X, U) \to (Y, V)$ be a uniformly perfect mapping of a uniform space (X, U) onto a uniform space (Y, V). Then, uniformly Menger's property is preserved both in the image and the preimage direction.*

Proposition 1. *The space of real numbers R with natural uniformity U_R is a uniformly Menger space.*

Proof. Let $(\alpha_n : n \in \mathbb{N}) \subset U_R$ be an arbitrary sequence of uniform covers and $\beta = \{(n-1, n+1) : n = 0, \pm 1, \pm 2, \dots\}$ be an open cover of the space (R, U_R). Consider the following construction: for $n = 0$, due to the compactness of $[-1, 1]$, from the cover α_1 select a finite subfamily $\alpha_1^0 \subset \alpha_1$ such that $(-1, 1) \subset [-1, 1] \subset \bigcup \alpha_1^0$, for $n = 1$ from the cover α_2 select a finite subfamily $\alpha_2^0 \subset \alpha_2$ such that $(0, 2) \subset [0, 2] \subset \alpha_2^0$, and for $n = -1$ from the cover α_3 select a finite subfamily $\alpha_3^0 \subset \alpha_3$, such that $(-2, 0) \subset [-2, 0] \subset \alpha_3^0$, etc. Continuing this process, get a sequence $(\alpha_n^0 : n \in \mathbb{N})$ of finite subfamilies. Since β is a cover of the space (R, U_R) and each element $(n-1, n+1) \in \beta$ is covered by some finite subfamily α_n^0, then the family $\bigcup\limits_{n\in\mathbb{N}} \alpha_n^0$ is a cover of the space (R, U_R). Therefore, the space (R, U_R) is uniformly Menger. □

Corollary 4. *The space of the rational numbers Q with the uniformity induced from the uniformity U_R is a uniformly Menger space. The unit interval* $(0, 1)$ *as a subspace of* (R, U_R) *is also a uniformly Menger space.*

Proof. It follows from the Proposition 1 and [3, Theorm 7]. □

Theorem 6. *A uniformly locally compact space (X, U) is a uniformly Menger space if and only if it is a pre-Lindelöf space.*

Proof. Necessity:

Let (X, U) be a uniformly locally compact uniformly Menger space. Then, from the Theorem 1, (X, U) is a pre-Lindelöf space.

Sufficiency:

Let (X, U) be a uniformly locally compact pre-Lindelöf space. Let us show that (X, U) is a uniformly Menger space. Let $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers and β a uniform cover consisting of compact subsets. Without loss of generality, assume that β is a countable uniform cover consisting of compact subsets, i.e., $\beta = \{B_1, B_2, \ldots, B_n, \ldots\}$. For any $n \in \mathbb{N}$, due to the compactness of B_n, there exists a finite subfamily $\alpha_n^0 \subset \alpha$ such that $B_n \subset \bigcup \alpha_n^0$. Since β is a cover of the space, the family $\bigcup\limits_{n\in\mathbb{N}} \alpha_n^0$ is also a cover of the space (X, U). Consequently, (X, U) is a uniformly Menger space. □

Proposition 2. *The completion of a uniformly Menger space is a uniformly Menger space.*

Proof. Let $(\tilde{X}, \tilde{U})$ be the completion of the uniformly Menger space (X, U) and $(\tilde{\alpha}_n : n \in \mathbb{N}) \subset \tilde{U}$ be an arbitrary sequence of uniform covers. Put $\alpha_n = \tilde{\alpha}_n \wedge \{X\}$. Then, from the definition of completion of uniform spaces $(\alpha_n : n \in \mathbb{N}) \subset U$. Since (X, U) is a uniformly Menger space, there exists a sequence $(\beta_n : n \in \mathbb{N})$ of finite subfamilies such that for any $n \in \mathbb{N}$, β_n is a subfamily of α_n and $\bigcup\limits_{n\in\mathbb{N}} \beta_n$ is a cover of the space (X, U). Then, there is a sequence $(\tilde{\beta}_n : n \in \mathbb{N})$ of finite subfamilies, such that for any $n \in \mathbb{N}$, $\tilde{\beta}_n$ is a subfamily of $\tilde{\alpha}_n$ and $\tilde{\beta}_n \wedge \{X\} = \beta_n$ for any $n \in \mathbb{N}$. It is easy to see that $\bigcup\limits_{n\in\mathbb{N}} \tilde{\beta}_n$ is a cover of the space $(\tilde{X}, \tilde{U})$. Consequently, $(\tilde{X}, \tilde{U})$ is a uniformly Menger space. □

Theorem 7. *The remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ of a uniform space (X, U) is a uniformly Menger space if and only if for any sequence $(\alpha_n : n \in \mathbb{N}) \subset U$ there exists a sequence $(\alpha_n^0 : n \in \mathbb{N})$ of finite subfamilies such that $\bigcup\limits_{n\in\mathbb{N}} \alpha_n^0$ is a co-cover of the uniform space (X, U).*

Proof. Necessity:

Let the remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ of a uniform space (X, U) be a uniformly Menger space and $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers of the space (X, U). Then $(\hat{\alpha}_n : n \in \mathbb{N}) \subset \tilde{U}_{\tilde{X} \backslash X}$, where $\hat{\alpha}_n = \tilde{\alpha}_n \wedge \{\tilde{X} \backslash X\}$, $\tilde{\alpha}_n = \{\tilde{A}_n : A_n \in \alpha_n\}$, $\tilde{A}_n = \tilde{X} \backslash [X \backslash A_n]_{\tilde{X}}$. Since $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ is uniformly Menger space, there exists a sequence $(\hat{\alpha}_n^0 : n \in \mathbb{N})$ of finite subfamilies such that $\bigcup\limits_{n\in\mathbb{N}} \hat{\alpha}_n^0$ is a cover of the space $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$.

Let $\alpha_n^0 = \tilde{\alpha}_n^0 \wedge \{X\}$. $\tilde{\alpha}_n^0$ is a finite subfamily, so α_n^0 is a finite subfamily. Let F be an arbitrary free Cauchy filter of the space (X, U). Then it converges to some point $\hat{x} \in \tilde{X} \backslash X$. There is $\hat{A} \in \bigcup\limits_{n \in \mathbb{N}} \hat{\alpha}_n^0$ such that $\hat{A} \ni \hat{x}$, $\hat{A} = \tilde{A} \bigcap (\tilde{X} \backslash X)$. Denote by $\tilde{B}(\hat{x})$ the filter of neighborhoods of the point $\hat{x}$. Note $\tilde{B}(\hat{x}) \bigcap X = F'$ and $F' \subset F$. Then $A \in F$. Therefore, $\bigcup\limits_{n \in \mathbb{N}} \alpha_n \bigcap F \neq \varnothing$, i.e., the family $\bigcup\limits_{n \in \mathbb{N}} \hat{\alpha}_n^0$ is a co-cover of the space (X, U).

Sufficiency:

Let $\{\hat{\alpha}_n\}$ be an arbitrary sequence of uniform covers of the space $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$. Then, there is a sequence $(\alpha_n : n \in \mathbb{N})$ of uniform covers of the space (X, U), such that $(\tilde{\alpha}_n^0 \wedge \{\tilde{X} \backslash X\} : n \in \mathbb{N}) = (\hat{\alpha}_{\mathrm{n}} : \mathrm{n} \in \mathbb{N})$. Hence, by the conditions of the theorem that there exists a sequence $(\alpha_n^0 : n \in \mathbb{N})$ of finite subfamilies such that the family $\bigcup\limits_{n \in \mathbb{N}} \alpha_n^0$ is a co-cover of the space (X, U). Put $\bigcup\limits_{n \in \mathbb{N}} \hat{\alpha}_n^0$, where $\hat{\alpha}_n^0 = \tilde{\alpha}_n^0 \bigcap (\tilde{X} \backslash X)$, $\tilde{\alpha}_n^0 = \{\tilde{A}_n : A_n \in \alpha_n^0\}$, $\tilde{A}_n = \tilde{X} \backslash [X \backslash A_n]_{\tilde{X}}$. Let us prove that the family $\bigcup\limits_{n \in \mathbb{N}} \hat{\alpha}_n^0$ is a cover of the space $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$. Let $\hat{x} \in \tilde{X} \backslash X$ be an arbitrary point. Denote by $\tilde{B}(\hat{x})$ the filter of neighborhoods of the point $\hat{x}$ in $(\tilde{X}, \tilde{U})$, i.e $\tilde{B}(\hat{x})$ is a minimal filter Cauchy of the completion $(\tilde{X}, \tilde{U})$. Put $F = \tilde{B}(\hat{x}) \bigcap X$. Then, it is easy to see that F is a free Cauchy filter of the space (X, U). From this it follows that $\bigcup\limits_{n \in \mathbb{N}} \alpha_n^0 \bigcap F \neq \varnothing$, i.e., there exists $A \in \bigcup\limits_{n \in \mathbb{N}} \alpha_n^0$ such that $A \in F$. It is clear that $F \in \tilde{A} \in \tilde{B}(\hat{x})$. Hence, $\hat{A} \ni \hat{x}$, $\hat{A} \in \bigcup\limits_{n \in \mathbb{N}} \hat{\alpha}_n^0$, $\hat{A} = \tilde{A} \bigcap (\tilde{X} \backslash X)$. Consequently, the family $\bigcup\limits_{n \in \mathbb{N}} \alpha_n^0$ is a cover of the space $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$.

Thus, the remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ is a uniformly Menger space. □

Proposition 3. *Any compact uniform space (X, U) is a uniformly Menger space.*

Proof. Let $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers. Then, due to the compactness of the space (X, U), for any $n \in \mathbb{N}$ the uniform cover α_n contains a finite subcovering. Let $(\alpha_n^0 : n \in \mathbb{N})$ be a sequence of such finite subcovers. Then $(\alpha_n^0 : n \in \mathbb{N})$ is the desired sequence of finite subfamilies. Therefore, (X, U) is a uniformly Menger space. □

Theorem 8. *The product $(X \times Y, U \times V)$ of a uniformly Menger space (X, U) and a precompact uniform space (Y, V) is a uniformly Menger space.*

Proof. Let $(X \times Y, U \times V)$ be the product of the uniformly Menger space (X, U) and the precompact space (Y, V), and let $(\gamma_n : n \in \mathbb{N}) \subset U \times V$ be an arbitrary sequence of uniform covers. Let $\gamma_n = \alpha_n \times \beta_n$, $\alpha_n \in U$, $\beta_n \in V$

for any $n \in \mathbb{N}$. Since the space (X, U) is uniformly Menger there exists a sequence $(\sigma_n : n \in \mathbb{N})$ such that for any $n \in \mathbb{N}$, σ_n is a finite subfamily of α_n and $\bigcup\limits_{n\in\mathbb{N}} \sigma_n$ is a cover of the space (X, U), and for any $n \in \mathbb{N}$ the cover β_n contains a finite subcover β_n^0. Then, the family $(\sigma_n \times \beta_n^0 : n \in \mathbb{N})$ is a finite subfamily of the cover $\alpha_n \times \beta_n$. Let us show that the family $\bigcup\limits_{n\in\mathbb{N}} \sigma_n \times \beta_n^0$ is a cover of the space $(X \times Y, U \times V)$. Let $(x, y) \in X \times Y$ be an arbitrary point. Then, there are $n^* \in \mathbb{N}$ and $C \in \sigma_{n^*}$, such that $x \in C$. Since β_n^0 is a finite subcover of the space (Y, V), then for any $n \in \mathbb{N}$, and so for n^*, there exists $B \in \beta_{n^*}^0$ such that $y \in B$. Hence, $(x, y) \in C \times B \in \sigma_{n^*} \times \beta_{n^*}^0$. Therefore, the space $(X \times Y, U \times V)$ is uniformly Menger. $\square$

Corollary 5. *The product of a uniformly Menger space and a compact uniform space is a uniformly Menger space.*

Theorem 9. *The finite discrete sum $(X, U) = \coprod\{(X_i, U_i) : i = 1, 2, \ldots, m\}$ of uniformly Menger spaces $(X_i, U_i), i = 1, 2, \ldots, m$ is uniformly Menger.*

Proof. Let $(X_i, U_i), i = 1, 2, \ldots, m$, be a uniformly Menger space. Let $(\alpha_n : n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers. Then, $(\alpha_{n,X_i} : n \in \mathbb{N}) \subset U_i, \alpha_{n,X_i} = \alpha_n \wedge \{X_i\},\ i = 1, 2, \ldots, m$. Therefore, for every $i \in \{1, 2, \ldots, m\}$ there is a sequence β_{n,X_i} such that for any $n \in \mathbb{N}$ the family β_{n,X_i} is finite and $\bigcup\limits_{n\in\mathbb{N}} \beta_{n,X_i}$ is a cover of the space (X_i, U_i). Put $\beta_n = \bigcup\limits_{i=1}^{n} \beta_{n,X_i}$. Since the spaces $(X_i, U_i), i = 1, 2, \ldots, m$ are pairwise disjoint in the space (X, U), then β_n is a finite subfamily for α_n. According to the definition of a discrete sum of uniform spaces, we have that the family $\bigcup\limits_{n\in\mathbb{N}} \beta_n$ is a cover of the space (X, U). Therefore, the uniform space (X, U) is uniformly Menger. $\square$

3. Conclusion

In this paper we have investigated some important properties of the uniformly Menger spaces which lie between precompact uniform spaces and pre-Lindelöf uniform spaces. Especially, we proved that under uniformly perfect mappings the uniformly Menger property is preserved, both in the image and the preimage direction. Our further research could include investigation of other classes of uniform spaces such as Rothberger and Hurewicz uniform spaces.

4. Conflict of interests

The authors declare that they have no conflict of interest.

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